On the Global Quantum Dynamics of Multi-Lattice Systems with Non-linear Classical Effects*

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The microscopic dynamics for a class of long range interacting multi-lattice quantum systems is constructed in the thermodynamical limit by means of operator algebraic concepts. By direct estimations the existence of the limiting Schrödinger dynamics is proven for a set of states, which comprises also globally non-equilibrium situations. The expectation values of the classical observables in the pure phase states are shown to satisfy a set of coupled non-linear differential equations. The limiting Heisenberg dynamics is derived as a W^*-automorphism group in the partially universal von Neumann algebra which corresponds to the selected set of states; it is in general, however, not σ-weakly continuous in the time parameter.

1. Introduction

The purpose of the present work is the rigorous construction of the microscopic dynamics for certain infinite quantum lattice systems with long range interacting forces. The lattice consists in general of finitely many sublattices, which may also be interpreted as spacially separated subsystems. Within one sublattice the interaction is chosen invariant under the permutation of the lattice indices. This simplification makes possible the complete and rigorous formulation of the microscopic and macroscopic quantum mechanics also for situations which differ globally from equilibrium, and still there are some physically relevant models in the considered class of systems, such as superconductors (with or without paramagnetic impurities), metamagnetic systems, metallic alloys and Josephson-junctions.

It is known that for long range interactions the dynamics in the thermodynamical limit cannot be realized as a C^*-dynamical automorphism group in the quasi-local algebra [1, 2]. Here our first step is to construct a limiting Schrödinger dynamics, which acts as a group of affine bijections in a convex norm-closed subset of the state space, which is invariant under local perturbations and is called the "folium of physical states" [2, 3]. The mathematical characterization of this folium is to be the smallest one which contains all permutation invariant states. Intuitively that means that its states are homogeneous outside the local lattice regions, so that the specific extensive observables take on well defined values. In this sense it constitutes a constructive realization of the set of "classical states" of Hepp and Lieb [17] and allows for a detailed elaboration of the microscopic quantum mechanics rather than giving the macroscopic classical aspects only.

The existence proofs for the Schrödinger dynamics rest on detailed estimations for the iterated local Heisenberg generators averaged over physical states. In the main text we emphasize the basic line of reasoning and defer the more technical arguments to the appendix. But even there we found it inadequate to carry out all required steps for the complete inductions, which run over the powers of the Heisenberg generators and over the cardinalities of the local lattice regions. Nevertheless, we hope to give an idea of the complexity of reasoning, which is necessary to *prove* the existence of a limiting dynamics.

The extreme boundary of the folium of physical states consists of factor states of the quasi-local algebra, which are interpreted as (non-equilibrium) pure phase states, a notion which has been inaccessible in usual many body physics.

Their macroscopic features may be expressed by a finite set of parameters which constitute a bridge to phenomenological (non-equilibrium) thermodynamics. These parameters may be chosen as the expectation values of the specific extensive observables (density observables), for which a self-consistent system of coupled non-linear differential equations is derived.

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The transition to the Heisenberg picture requires an extension of the quasi-local algebra. Here it is given by the folium of physical states in form of the associated partially universal von Neumann algebra. Using some results of the general theory of Jordan-automorphisms [4–6] we arrive at a group of W^*-automorphisms as the dual mappings of the Schrödinger dynamical transformations. We prove that the Heisenberg dynamics is not σ-weakly continuous in the time parameter. Therefore, the notion of a global energy as the generator of the time translations would require some modifications of the formalism. From [6] one may guess that the set of physical states has again to be diminished.

Altogether we can say that already the present model discussion demonstrates that the microscopic interaction combined with a selected set of appropriate states uniquely determines the dynamics of a global quantum description. In treatable models of physical interest the usual procedure of "guessing" non-equilibrium equations thus can be replaced by a systematical derivation from the microscopic first principles.

2. Algebra of Observables, Microscopic Interaction Ansatz, and the Folium of Physical States

In this section we give the notations and definitions of the basic algebras, observables, model Hamiltonians and states. We use $r \in \mathbb{N}$ for the number of different sublattices, and each sublattice q, $1 \leqq q \leqq r$, will be identified with the set Z of integers. (Within every sublattice we will make a permutation invariant interaction ansatz, and thus the dimensions of the sublattices themselves are irrelevant.) For the entire lattice we write

$$K := \mathbb{Z} \times \{1, \ldots, r\},$$

the Cartesian product set of $\mathbb{Z}$ and $\{1, \ldots, r\}$.

An element $k \in K$, $k = (i, q)$ with $i \in \mathbb{Z}$ and $1 \leqq q \leqq r$, designates then the lattice point i in the sublattice q. For every finite region $\Lambda \subseteq K$ there are uniquely defined subsets $\Lambda(q) \subset K$:

$$\Lambda(q) := \{(i, q') \in \Lambda;\ q' = q\},$$

and we have the decomposition

$$\Lambda = \bigcup_{q=1}^{r} \Lambda(q).$$

For a simplification of notations let the sets $\Lambda(q)$ and $\{i \in \mathbb{Z}, (i, q) \in \Lambda(q)\}$ be identified.

On the set of finite lattice regions there is a canonical order relation defined by the usual set inclusion.

Furthermore, for every $\Lambda \subset K$ we define the cardinality

$$|\Lambda| := \sum_{q=1}^{r} |\Lambda(q)| := \sum_{q=1}^{r} \sum_{i \in \Lambda(q)} 1.$$

With these definitions we can form the directed set of local lattice regions:

$$I := \{\Lambda \subseteq K;\ |\Lambda| < \infty\}.$$

In every sublattice there may be a different kind of spin particles. Therefore we have for any $q \in \{1, \ldots, r\}$ an integer $n(q) \in N$, giving the dimension of the one particle Hilbert space

$$H(q) := \mathbb{C}^{n(q)},$$

and we get the associated algebra of observables

$$A_k := A_{(i,q)} \cong B(\mathbb{C}^{n(q)}) =: B^q$$

for every $k \in K$, which is isomorphic to the complex $n(q) \times n(q)$ matrix algebra. In every lattice region $\Lambda \in I$ the local algebra of observables A_Λ is defined by the tensorial product of the A_k's:

$$A_\Lambda := \bigotimes_{k \in \Lambda} A_k = \bigotimes_{q=1}^{r} \bigotimes_{i \in \Lambda(q)} A_{(i,q)}.$$

For the infinite lattice there is now a canonical procedure, the C^*-inductive limit [7, 8], to get a suitable algebra of observables into which all local ones are canonically embedded. This C^*-algebra, called the quasilocal C^*-algebra of the multilattice spin system, can be written as

$$A := \overline{\bigcup_{\Lambda \in I} A_\Lambda}^{\|\cdot\|} = \bigotimes_{k \in K} A_k = \bigotimes_{q=1}^{r} \left(\bigotimes_{i \in \mathbb{Z}} A_{(i,q)} \right).$$

In every one-particle algebra B^q, $q \in \{1, \ldots, r\}$, we define an arbitrarily choosen (but from now on fixed) basis of linear, selfadjoint and normalized operators:

$$\{e_{l,q} \in B(C^{n(q)});\ \|e_{l,q}\| = 1,\ e^*_{l,q} = e_{l,q},\ l = 1, \ldots, n(q)^2\};$$

and for every lattice point $k = (i, u) \in K$ we get by the definition

$$\pi_k(e_{l,q}) := \delta_{u,q} \cdot \ldots \mathbb{1} \otimes \underbrace{e_{l,q}}_{k=(i,q)} \otimes \mathbb{1} \ldots$$

$$= \delta_{u,q}\, \pi_{(i,q)}(e_{l,q})$$

an embedding morphism

$$\pi_k\colon B \to A \quad \text{with} \quad B := \bigoplus_{q=1}^{r} B^q$$

the composite unit cell algebra. We also see that

$$\pi_{(i,q)}: B^q \to A_{(i,q)}$$

is a *-isomorphism onto the one-particle algebra $A_{(i,q)}$.

Now let us define the basic local mean field operators.

Definition 1:

For every unit cell operator $x \in B$ and local region $\Lambda \in I$ we define the local mean field operator $m_\Lambda(x) \in A_\Lambda$

$$m_\Lambda(x) := (1/|\Lambda|) \sum_{k \in \Lambda} \pi_k(x).$$

In mean field theories the energy of the system does not depend on the shape of the local region, although there is the possibility for coupled mean field systems to prescribe the volume proportions between the sublattices in the infinite volume limit. The total energy of finite lattice sectors will depend on these concentration parameters, and to get a general notation we define so-called volume functions. Let be

$$V: I \to \mathbb{C}$$

a map of the index set I, then V is called a volume function if there is a polynomial $P: \mathbb{R}^r \to \mathbb{C}$ such that for all $\Lambda \in I$ it holds:

$$V(\Lambda) = P(|\Lambda|/|\Lambda(1)|, \ldots, |\Lambda|/|\Lambda(r)|).$$

A vector $\boldsymbol{c} = (c_1, \ldots, c_r) \in \mathbb{R}^r$ is called a concentration (for the sublattices) if $c_q > 0$, $q \in \{1, \ldots, r\}$ and $\sum_{q=1}^{r} c_q = 1$.

Let be $\tilde{I} \subseteq I$ a directed and absorbing index subset, then $I(c) := \tilde{I}$ is called an index subset with respect to the concentration c, iff for every $\varepsilon > 0$ there is some finite region $\Lambda_0 \in I(c)$ such that for every $q \in \{1, \ldots, r\}$ and all $\Lambda_0 \subseteqq \Lambda \in I(c)$ the following inequality is valid:

$$||\Lambda|/|\Lambda(q)| - c_q| \leqq \varepsilon.$$

Now we are able to give the exact definition of the Hamiltonian for every finite lattice region.

Definition 2:

For every $\Lambda \in I$ the Hamiltonian of the system in Λ is given by the operator $h_\Lambda \in A_\Lambda$:

$$h_\Lambda := |\Lambda| \cdot \left\{ \sum_{i=1}^{r} \sum_{k \in \nu(i)} a(i,k)\, m_\Lambda(e_{i,k}) + \sum_{i,j=1}^{r} \sum_{k \in \nu(i),\, n \in \nu(j)} V(i,k,j,n,\Lambda)\, m_\Lambda(e_{k,i})\, m_\Lambda(e_{n,j}) \right\}$$

with $a(i,k) \in \mathbb{R}$ and volume functions $V(i,k,j,n,\Lambda) = V(j,n,i,k,\Lambda)$ for all $k \in \nu(i)$ and $n \in \nu(j)$, $\nu(i) := \{1, \ldots, n(i)^2\}$ and all $i, j \in \{1, \ldots, r\}$.

In the next step the folium of relevant physical states of the infinite lattice will be constructed with the help of the permutation group. The r-dimensional permutations restricted to local regions give the characteristic symmetry of the interactions in definition 2. This symmetry is extented to global permutation invariance in the infinite volume limit [9, 10]. We say $p: \Lambda \to \Lambda$ is an r-dimensional permutation in a region $\Lambda \in I$ if there is a vector $\boldsymbol{p} = (p_1, \ldots, p_r)$ of ordinary permutations within the integers $\mathbb{Z}$ with $p_1(\Lambda(1)) = \Lambda(1), \ldots, p_r(\Lambda(r)) = \Lambda(r)$. We denote the group of all finite r-dimensional permutations on the lattice $\mathbb{Z} \times (1, \ldots, r\}$ with the symbol P.

Let be $a = \bigotimes_{q=1}^{r} \bigotimes_{i \in \Lambda(q)} a_{(i,q)} \in A$ a product operator. Then the extension of the map

$$\alpha_p(a) := \bigotimes_{q=1}^{r} \bigotimes_{i \in \Lambda(q)} a_{(p(i),q)}, \quad p \in P,$$

implements the permutation group P as a C^*-automorphism group on the quasilocal algebra A. The group $\{\alpha_p\}_{p \in P}$ is asymptotic abelian (cf. e.g. [8]).

Let be S $(= A^*_{+1})$ the state space of A, then the convex set $S^P \subset S$:

$$S^P := \{\psi \in S; \langle \psi; \alpha_p(a) \rangle = \langle \psi; a \rangle, \forall p \in P, \forall a \in A\}$$

is called the set of permutation invariant states on A, and with $\mathrm{d}S^P$ we denote the extremal boundary of S^P.

The local perturbations of the states in S^P will generate our folium F of relevant system states, and on this subset of the whole state spaces we will get weak convergence of the local Schrödinger dynamics $\{\alpha^*_{\Lambda,t}\}_{t \in \mathbb{R}}$, where the $\alpha^*_{\Lambda,t}: S \to S$ are the adjoint maps to the local Heisenberg automorphisms:

$$\alpha_{\Lambda,t}(a) := \exp(i t h_\Lambda)\, a \exp(-i t h_\Lambda), \quad a \in A.$$

Definition 3:

We call the convex set $F \subset S$:

$F :=$ convex hull $\{\psi \in S;$ there is an operator $a \in A$ and a permutation invariant state $\omega \in S^P$ such that for all $b \in A$ it holds: $\langle \psi; b \rangle = \langle \omega; a^* b a \rangle\}^{-\|\cdot\|}$

the folium of permutation invariant states.

The folium F is large enough to describe all microscopic and macroscopic features of the whole class of mean field models. For example, all the limiting Gibbs states and their local perturbations for arbitrary temperatures of our model class are in F. But we have there also states which deviate from equilibrium with respect to infinitely many degrees of freedom. We say that F is the physically relevant state space for mean field models [11].

Now let us look at the observables in A. If we define classical observables as those which commute with all other operators in A, then only $\mathbb{C} \cdot 1$, the trivial ones, are in A. On the other hand one knows from the phenomenological theory that for a mean field system there are in general interesting macro-observables which are not equal to c-numbers. These we get only by constructing an extended algebra. We start with the partially universal representation $\{\pi_F, H_F\}$ of A with respect to the folium F:

$$\pi_F := \bigoplus_{\omega \in F} \pi_\omega, \quad H_F := \bigoplus_{\omega \in F} H_\omega,$$

with the GNS-representations $\{\pi_\omega, H_\omega, \Omega_\omega\}$ for every $\omega \in F$. It is known that for every $\omega \in F$ there exists a central projection $S_\omega \in Z_F$ $(:= \pi_F(A)'' \cap \pi_F(A)')$ in the von Neumann algebra Z_F (unique up to quasi-equivalence) which is defined by the central support of the state $\omega \in F$. If $\psi, \omega \in F$ are disjoint states (and there are overcountably many of them in F), then we have $S_\omega S_\psi = S_\psi S_\omega = 0$, and by this we get many nontrivial classical observables in $M_F = \pi_F(A)''$ [12]. We consider the von Neumann algebra M_F as the relevant set of observables for the global quantum mechanics of the mean field theories. For factorial states $\omega \in F$ the projection S_ω is minimal in Z_F, which can be physically interpreted as a maximal classical determination of the infinite lattice system, i.e. a pure phase of the system [13]. Another important part of Z_F we get by the infinite volume limit in the strong operator topology (=: s_-) of the represented mean field operators from definition 1. This convergence will now be examined in general.

Definition 4:

Let be $i, j \in \mathbb{N}$, $0 \leqq q \leqq i$, $1 \leqq k \leqq j$, $V_{q,k}(\Lambda_{q,k})$ volume functions, $a_{q,k} \in A$, $x_k \in B$ and $n(q,k) \in \mathbb{N}_0$ arbitrary. We define

$$\Gamma(\Lambda_{1,1}, \ldots, \Lambda_{i,j}; x_1, \ldots, x_j) := \sum_{q=0}^{i} \prod_{k=1}^{j} \{a_{q,k} V_{q,k}(\Lambda_{q,k})\, m_{\Lambda_{q,k}}(x_k)^{n,(q,k)}\}$$

for a generalized mean field polynomial of the degree

$$\deg(\Gamma) := \max_{q=0,\ldots,i} \left(\sum_{k=1}^{j} n(q,k) \right).$$

For every map $p: \{(1,1), \ldots, (i,j)\} \to \{(1,1), \ldots, (i,j)\}$ we define

$$p \circ \Gamma := \sum_{q=0}^{i} \prod_{k=1}^{j} (a_{q,k} V_{p(q,k)}(\Lambda_{p(q,k)})\, m_{\Lambda_{p(q,k)}}(x_k)^{n(q,k)}).$$

In the following proposition we use quasi-contained representations. We say that a representation $\{\eta, X\}$ of A on a Hilbert space X is quasi-contained in a representation $\{\pi, H\}$ of A, iff there is a normal extension of the representation $\{\eta, X\}$ to a representation of the von Neumann algebra $M_\pi := \pi(A)''$.

Proposition 5:

Let be Γ a generalized mean field polynomial, $c \in \mathbb{R}^r$ a concentration, $I(c)$ an index subset for the concentration c and $p: \{(1,1), \ldots, (i,j)\} \to \{(1,1), \ldots, (i,j)\}$ an arbitrary map. Then we get for every representation $\{\eta, X\}$ of A quasi-contained in $\{\pi_F, H_F\}$ the following limits:

$$\begin{aligned}
&\Gamma_\eta(c, \ldots, c; m_\eta(x_1(c)), \ldots, m_\eta(x_j(c))) \\
&:= s-\lim_{\Lambda_{1,1} \in I(c)} \ldots \\
&\qquad \ldots s-\lim_{\Lambda_{i,j} \in I(c)} \eta(\Gamma(\Lambda_{1,1}, \ldots, \Lambda_{i,j}; x_1, \ldots, x_j)) \\
&= s-\lim_{\Lambda_{1,1} \in I(c)} \ldots s-\lim_{\Lambda_{i,j} \in I(c)} \eta(p \circ \Gamma) \\
&= s-\lim_{\Lambda \in I(c)} \eta(\Gamma(\Lambda, \ldots, \Lambda; x_1, \ldots, x_j)) \\
&= \sum_{q=0}^{i} \prod_{k=1}^{j} (\eta(a_{q,k})\, V_{q,k}(c)\, m_\eta(x_k, c)^{n(q,k)}),
\end{aligned}$$

where we have defined the basic mean field operators

$$m_\eta(x_k, c) := s-\lim_{\Lambda \in I(c)} \eta(m_\Lambda(x_k)) \in Z_\eta$$

with $Z_\eta = \pi_\eta(A)'' \cap \pi_\eta(A)'$ the center of $\pi_\eta(A)''$ and the limit volume functions

$$V_{q,k}(c) := \lim_{\Lambda \in I(c)} V_{q,k}(|\Lambda|/|\Lambda(1)|, \ldots, |\Lambda|/|\Lambda(r)|).$$

If furthermore the coefficients $a_{q,k}$ are complex numbers, then $\Gamma_\eta(c, \ldots, c; m_\eta(x_1, c) \ldots) \in Z_\eta$, i.e. it is a classical observable in the representation $\{\eta, X\}$.

Proof: The convergence of the mean field operators

$$m_{\pi_\omega}(x, c) := s-\lim_{\Lambda \in I(c)} \pi_\omega(m_\Lambda(x)) \in Z_{\pi_\omega}$$

for any state $\omega \in F$ is proved by a very slight variation of proposition 2.2 in [14]. From this we get the convergence of the mean field operators in the partially universal representation $\{\pi_F, H_F\}$. Because of the normal extension property of $\{\eta, X\}$ we have shown the existence of the limits

$$m_\eta(x, c) = s - \lim_{\Lambda \in I(c)} \pi_\eta(m_\Lambda(x)) \in Z_\eta \quad \text{for all } x \in B.$$

Together with the uniform boundedness of the generalized polynomials with respect to $\Lambda \in I(c)$ and the continuity of the operator product with respect to the strong operator topology on bounded sets, this completes the proof.

3. Infinite Volume Limit of the Local Dynamics

The convergence properties of the multiple Liouville operators

$$L_\Lambda^n\colon A \to A, \quad n \in \mathbb{N}_0$$

defined by the local Hamiltonians

$$L_\Lambda^n(a) := [h_\Lambda, \ldots, [h_\Lambda, a] \ldots], \quad a \in A$$

determine the possibility of a global dynamics for the infinite lattice. In our case we cannot expect convergence within the quasilocal algebra A because we have already for an arbitrary single cell operator $x \in B$ at the lattice point $k \in K$ the equation

$$\begin{aligned} L_\Lambda^1(\pi_k(x)) = &\sum_{i=1}^{r} \sum_{u \in v(i)} a_{i,u}\, \pi_k([e_{u,i}, x]) \\ &+ \sum_{i,j=1}^{r} \sum_{u \in v(i),\, v \in v(j)} V(i, u, j, v, \Lambda) \\ &\cdot \{m_\Lambda(e_{u,i}) \cdot \pi_k([e_{v,j}, x]) + \pi_k([e_{u,i}, x])\, m_\Lambda(e_{v,j})\}. \end{aligned}$$

This simple commutator constitutes just a nontrivial meanfield polynomial which only converges in certain representations of A with respect to the strong operator topology.

Another problem for the construction of a limiting dynamics – besides the convergence of the local Liouville generators – is to find some uniform boundedness of these with respect to $\Lambda \in I$. These properties will be shown in the following, where special care is devoted to the inclusion relations for the lattice regions of the type $\chi \subseteq \varkappa \subseteq \Lambda \in I$. These are very important and in most cases cannot be removed.

Lemma 6:

Let be $c \in \mathbb{R}^r$ a concentration and $I(c)$ a proper index subset, then there is some $M(c) < \infty$ and $\Lambda_0 \in I(c)$ such that for all $\Lambda_0 \subseteqq \Lambda \in I(c)$ it holds:

$$\sum_{i=1}^{r} \sum_{u \in v(i)} 2\,|a(i, u)| + \sum_{i,j=1}^{r} \sum_{u \in v(i),\, v \in v(j)} 4\,|V(i, u, j, v, \Lambda)| \leqq M(c).$$

We call a region $\Lambda \in I(c)$ large, whenever this inequality holds. Furthermore, let be $i, j \in N$, $\lambda := \{(\lambda_1, 1), \ldots, (\lambda_r, r)\} \in K$ an arbitrary set of r lattice points, then for all subsectors $\lambda \subseteq \chi \subseteq \varkappa \subseteq \Lambda \in I(c)$ we get

i) for every unit cell operator $x \in B$:

$$\begin{aligned} \| L_\Lambda^i(L_\varkappa^j(m_\chi(x)))\| &\leqq \sum_{q=1}^{r} (|\chi(q)|/|\chi|)\, \| L_\Lambda^i(L_\varkappa^j(\Pi_{(\lambda_q, q)}(x)))\| \\ &\leqq \|x\|_\infty (i+j)!\, M(c)^{i+j}, \end{aligned}$$

where we write $\|x\|_\infty := \max_{q=1,\ldots,r} \|x_q\|$;

ii) for every χ-local operator $a \in A_\chi$:

$$\| L_\Lambda^i(L_\varkappa^j(a))\| \leqq \|a\|\, (i+j)!\, (|\chi|\, M(c))^{i+j}.$$

Proof: See appendix!

Lemma 7:

Let be $c \in \mathbb{R}^r$ a concentration, $i, j, u, v \in \mathbb{N}$, Γ a generalized mean field polynomial with strictly local coefficient operators (i.e. there is some $\Lambda_0 \in I$ with $a_{q,s} \in A_{\Lambda_0}$ for all $0 \leqq q \leqq i$ and $1 \leqq s \leqq j$) and an arbitrary map p on the set $\{(1, 1), \ldots, (i, j)\}$. Then for every representation $\{\eta, X\}$ of A, quasi-contained in $\{\pi_F, H_F\}$, it holds:

$$\begin{aligned} &L_\eta(u, v, \Gamma(c, \ldots, c; m_\eta(x_1, c), \ldots, m_\eta(x_j, c))) \\ &:= s - \lim_{\Lambda_{1,1} \in I(c)} \ldots s - \lim_{\Lambda_{i,j} \in I(c)}\; s - \lim_{\varkappa \in I(c)}\; s - \lim_{\Lambda \in I(c)} \\ &\quad \eta(L_\Lambda^u(L_\varkappa^v(p \circ \Gamma(\Lambda_{1,1}, \ldots, \Lambda_{i,j}; x_1, \ldots, x_j)))) \\ &= s - \lim_{\Lambda_{1,1} \in I(c)} \ldots s - \lim_{\Lambda_{i,j} \in I(c)}\; s - \lim_{\Lambda \in I(c)} \\ &\quad \eta(L_\Lambda^{u+v}(p \circ \Gamma(\Lambda_{1,1}, \ldots, \Lambda_{i,j}; x_1, \ldots, x_j))) \\ &= s - \lim_{\Lambda \in I(c)} \eta(L_\Lambda^{u+v}(\Gamma(\Lambda, \ldots, \Lambda; x_1, \ldots, x_j))), \end{aligned}$$

with $L_\eta(\ldots) \in \eta(A)''$. If $a_{q,s} \in \mathbb{C}\mathbb{1}$ for all q and s, then we get $L_\eta \in Z_\eta$.

Proof: See appendix!

The limits $L_\eta(u, v, \ldots)$ of the local commutators are for $u+v=1$ *-derivations on $\eta(A)$ with $D(L_\eta(1, 0, \ldots)) = \eta(A)$. But we have already seen that $\eta(A)$ is not invariant under this mapping, and therefore it is not possible to define powers of this *-derivation. One can show that iff $L_\eta(1, 0, \ldots)$ is $\sigma(M_\eta, M_{\eta,*})$-closable to the generator of a W^*-automorphism group on M_η $(:= \pi_\eta(A)'')$ then the equation

$$L_\eta(u, v, \ldots) = L_\eta(1, 0, \ldots)^{u+v}$$

holds. If there is some state $\omega \in F$ with $\{\eta, X\} = \{\pi_\omega, H_\omega, \Omega_\omega\}$ which is invariant (i.e. $\lim_{\Lambda \in I(c)} \langle \omega; \alpha_{\Lambda,t}(a)\rangle = \langle \omega; a\rangle$ for all $a \in A$), then $L_{\pi_\omega}(1,0,\ldots)$ is $\sigma(M_{\pi_\omega}, M_{\pi_\omega,*})$-closable. In this case the set

$$\{\Gamma(c, \ldots, c; m_\omega(x_1, c), \ldots, m_\omega(x_j, c)); a_{q,s} \in \bigcup_{\Lambda \in I} A_\Lambda \text{ and } x_1, \ldots, x_j \in B\}$$

is analytic for the generator $L_{\pi_\omega}(1, 0, \ldots)$, and the W^*-automorphism group

$$\alpha_{\omega,t}\colon \; M_{\pi_\omega} \to M_{\pi_\omega}, \quad t \in \mathbb{R},$$

$$\alpha_{\omega,t} := \exp(i t L_{\pi_\omega}(1, 0, \ldots)),$$

is then the Heisenberg dynamics of the infinite lattice corresponding to the invariant state $\omega \in F$. This reconstruction of the Heisenberg dynamics depends essentially on the cyclic invariant vector Ω_ω. It is in general not possible to do anything similar for non-cyclic invariant representations [15] like the partially universal representation $\{\pi_F, H_F\}$ which will be studied in the following.

From the lemmas 6 and 7 we get the convergence of the time-transformed generalized mean field polynomials.

Theorem 8:

Let be the assumptions as in the previous lemmas, then the following limits exist in M_η for all $t, s \in \mathbb{R}$:

$$s-\lim_{\Lambda \in I(c)} \eta(\alpha_{\Lambda,t+s}(\Gamma(\Lambda, \ldots, \Lambda; x_1, \ldots, x_j)))$$

$$= s-\lim_{\varkappa \in I(c)} s-\lim_{\Lambda \in I(c)} \eta(\alpha_{\Lambda,t+s}(\Gamma(\varkappa, \ldots, \varkappa; x_1, \ldots, x_j)))$$

$$= s-\lim_{\Lambda_{1,1} \in I(c)} \ldots s-\lim_{\Lambda_{i,j} \in I(c)} s-\lim_{\varkappa \in I(c)} s-\lim_{\Lambda \in I(c)}$$

$$\eta(\alpha_{\Lambda,t}(\alpha_{\varkappa,s}(p \circ \Gamma(\Lambda_{1,1}, \ldots, \Lambda_{i,j}; x_1, \ldots, x_j)))).$$

Proof: See appendix!

A consequence of the proof to theorem 8 is the following corollary concerning the differentiation of the many time Greens functions. This corollary leads to results which have already been used in mean field theories, e.g. in [16] and [17].

Corollary 9:

Let be $\chi \in I$ and $a \in A_\chi$, $x \in B$, then for every state $\omega \in F$ and for all $b, d \in A$ the infinite volume Greens functions

i) $\lim_{\Lambda \in I(c)} \langle \omega; b\, \alpha_{\Lambda,t}(m_\Lambda(x))\, d\rangle$,

ii) $\lim_{\Lambda \in I(c)} \langle \omega; b\, \alpha_{\Lambda,t}(a)\, d\rangle$

are continuously differentiable, and the infinite volume limit may be interchanged with differentiation.

Proof: Let be $M(c)$ the constant of lemma 6.

For every $a \in A_\chi$ the infinite volume limit is uniform in the complex time $z \in \mathbb{C}$ if $|\mathrm{Im}(z)| < (|\varkappa|\, M(c))^{-1}$. Therefore we get from the convergence relations of theorem 8 a uniformly convergent net of analytic functions. For every index subset $I(c)$ there is an absorbing *sequence* of regions, and by the theory of analytic functions we arrive at the assertion.

4. Global Schrödinger Dynamics with Nonlinear Classical Effects

With the convergence relations of the last section a group of affine and invertible transformations on the folium F will be formed. The nontrivial classical part of this infinite volume dynamics will be described by a coupled system of nonlinear differential equations, if the dynamics is restricted to the invariant set of pure phase states of the folium F. These equations will be written down explicitly.

We always use the assumptions of the lemmas in section 3 without repeating them.

Theorem 10:

On the folium F there exists a family of affine maps $\{v_{c,t}\}_{t \in \mathbb{R}}$ with the following properties:

i) $v_{c,t}\colon \; F \to F, \quad \forall t \in \mathbb{R}$;

ii) $\nu_{c,t}(\nu_{c,s}(\omega)) = \nu_{c,t+s}(\omega)$, $\nu_{c,0}(\omega) = \omega$, $t, s \in \mathbb{R}$ and $\omega \in F$;

iii) $\mathbb{R} \ni t \to \langle \nu_{c,t}(\omega); a \rangle$ is continuous for all $\omega \in F$ and $a \in A$;

iv) $F \ni \omega \to \nu_{c,t}(\omega)$ is norm-continuous for every $t \in \mathbb{R}$;

v) $\langle \nu_{c,t}(\omega); a \rangle = \lim_{\Lambda \in I(c)} \langle \omega; \alpha_{\Lambda,t}(a) \rangle$, $\omega \in F$, $a \in A$ and $t \in \mathbb{R}$.

The group $\{\nu_{c,t}\}_{t \in \mathbb{R}}$ is called the global Schrödinger dynamics for the infinite lattice system with respect to the net of local mean field interactions $\{h_\Lambda\}_{\Lambda \in I}$ and concentration c.

Proof: See appendix!

Corollary 11:

It holds

i) $\nu_{c,t}(S^P) = S^P$;

ii) $\nu_{c,t}(\mathrm{d}S^P) = \mathrm{d}S^P$, for all $t \in \mathbb{R}$.

Proof: From the proof of theorem 10 we get i), and since $\mathrm{d}S^P$ is the extremal boundary of S^P and $\nu_{c,t}$ is an affine bijection by theorem 10ii), we get ii).

Let us define compact "parameter" sets $W(q)$, $1 \leqq q \leqq r$,

$$W(q) := \Big\{ m \in \mathbb{R}^{n(q)^2};\ \sum_{i \in \nu(q)} m_i\, e_{i,q} \geqq 0 \quad \text{and} \quad \mathrm{tr}_{\mathbb{C}^{n(q)}}\Big(\sum_i m_i\, e_{i,q}\Big) = 1 \Big\}.$$

The set $W(q)$ is a parametrization of the state space of the single lattice point algebra B^q with respect to the fixed linear basis $\{e_{i,q} \in B^q, i \in \nu(q)\}$. It is well known (cf. e.g. [8]) that the states in $\mathrm{d}S^P$ are infinite products of unit cell states, and therefrom the set

$$W := \mathop{\mathsf{X}}_{q=1}^{r} W(q)$$

represents a parametrization of $\mathrm{d}S^P$.

Let us remark that all factorial states in F are local perturbations of states in $\mathrm{d}S^P$.

Lemma 12:

The set of permutation invariant states S^P forms a Bauer simplex whose extremal points $\mathrm{d}S^P$ are just the product (and factorial) states. A state $\omega \in \mathrm{d}S^P$ is completely determined by the expectation values

$$m(\omega, e_{i,q}) := \langle \omega, \pi_{(j,q)}(e_{i,q}) \rangle = \lim_{\Lambda \in I(c)} (1/c_q) \langle \omega; m_\Lambda(e_{i,q}) \rangle$$

for any $j \in \mathbb{Z}$, every concentration $c \in \mathbb{R}^r$ and all $i \in \nu(q)$, $1 \leqq q \leqq r$. The parametrization map

$$\mathbb{P}\colon \mathrm{d}S^P \to W,$$

$$\mathbb{P}(\omega) := (m(\omega, e_{i,q});\ i \in \nu(q),\ 1 \leqq q \leqq r)$$

is an invertible map onto the compact set W.

Proof: The lemma is proven for a single lattice in [9], for the sublattice structure the proof is analogous because of the adequate definition of our permutation group P.

To every concentration $c \in \mathbb{R}^r$ we define the functions

$$f\colon W \to \mathop{\times}_{q=1}^{r} \mathbb{R}^{n(q)^2}$$

by

$$f_{j,q}(m, c) := \lim_{\Lambda \in I(c)} \langle \mathbb{P}^{-1}(m); (i/c_q)\,[h_\Lambda, m_\Lambda(e_{j,q})] \rangle$$

for all $j \in \nu(q)$, $1 \leqq q \leqq r$ and all $m \in W$. Then the following result determines completely the action of the global Schrödinger dynamics on the pure phase states $\mathrm{d}S^P$.

Theorem 13:

For every fixed concentration $c \in \mathbb{R}^r$ the time evolution of a state $\omega \in \mathrm{d}S^P$ at time $t \in \mathbb{R}$ is given by the state $\mathbb{P}^{-1}(\mathrm{m}(\mathrm{t}))$, where $m(t)$ is the unique solution of the nonlinear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} m = f(m, c), \quad m(0) = \mathbb{P}(\omega)$$

at time t and $f(m, c)$ is given by its components as defined above.

Proof: See appendix!

The time evolution of mixed phase states, like $\psi \in S^P/\mathrm{d}S^P$, may be determined by the central decomposition and the time evolution of the states from $\mathrm{d}S^P$ in the central support of ψ. Two states $\omega, \psi \in \mathrm{d}S^P$ are macroscopically different or disjoint iff their parametrization is different, that is whenever the matrices $\mathbb{P}(\omega) \neq \mathbb{P}(\psi)$ are unequal. Therefore the mean field system has nonequilibrium states whenever the function $f(m, c)$ is not the zero function on W. In this case there is always a continuous solution $m(t)$ of the differential equation which is not constant in time. In consequence the global Schrödinger dynamics transforms for every infinitely small time range between disjoint states. This is the reason for the discontinuity of the adjoint global Heisenberg dynamics. In detail the following result can be derived from theorem 10.

Theorem 14:

Let be $\{\nu_{c,t}\}_{t\in\mathbb{R}}$ the infinite volume Schrödinger dynamics, then there exists a family of W^*-automorphisms

$$\{\alpha_{c,t}: M_F \to M_F; t \in \mathbb{R}\}$$

of the partially universal von Neumann algebra M_F with the following properties:

i) for arbitrary $t \in \mathbb{R}$, $\omega \in F$, and $a \in M_F$ it holds:
$$\langle j(\omega); \alpha_{c,t}(a)\rangle = \langle j(\nu_{c,t}(\omega)); a\rangle;$$

ii) for arbitrary $t, s, \in \mathbb{R}$ and $a \in M_F$ it holds:
$$\alpha_{c,t}(\alpha_{c,s}(a)) = \alpha_{c,t+s}(a);$$

iii) for every generalized mean field polynomial and all $t \in \mathbb{R}$ it holds:
$$\alpha_{c,t}(\Gamma(c, \ldots, c; m_{\pi_F}(x_1, c), \ldots, m_{\pi_F}(x_j; c))) = s - \lim_{\Lambda \in I(c)} \pi_F(\alpha_{\Lambda,t}(\Gamma(\Lambda, \ldots, \Lambda; x_1, \ldots, x_j)));$$

iv) the map $t \to \alpha_{c,t}(a)$, $a \in M_F$, is in general (i.e. if $f(m, c) \neq 0$ for some $m \in W$) not $\sigma(M_F, M_{F,*})$-continuous.

Here we used again the symbol $j: F \to M_{F,*,+1}$ for the canonical embedding of the folium onto the normal states of M_F.

Proof: The family $\{j \cdot \nu_{c,t}\}_{t\in\mathbb{R}}$ is a dynamical transformation on $M_{F,*,+1}$ in the sense of [11], and we get from this reference the existence of the family of W^*-automorphisms $\{\alpha_{c,t}\}_{t\in\mathbb{R}}$ on M_F. From theorem 10 and the convergence properties of theorem 8, i) to iii) are easily derived. Result iv) is proven by the remarks above.

5. Conclusions

Having presented the detailed reasoning for constructing the global limiting dynamics for a class of quantum-lattice systems let us further discuss the results and compare them with the literature.

The conceptually important aspect of the considered quantum dynamics is, that it combines local and macroscopic features in equilibrium as well as in states very far from equilibrium. It also demonstrates, in how far the global dynamical aspects can be derived from the local microscopic interactions between the constituting particles. This ambitious goal has been achieved by the restriction to a simple model class and by the use of non-elegant estimation techniques. Concerning the model class we have already mentioned in the introduction that it comprises systems of physical relevance with interesting collective phenomena. The multi-lattice structure of our formalism enables us to couple finitely many of them. This is for example important for superconductor models, where a single lattice corresponds to a countable set of momentum vectors (cf. e.g. [22] and references therein). Two coupled superconductors from a Josephson junction, the operator-algebraic treatment of which has been attempted in [23]. Two Josephson junctions are combined to a SQUID, and even more complex macroscopic arrays of Josephson junctions have been fabricated [24]. An operator algebraic treatment of multi-lattice system in position space as a model for metallic alloys has been worked out in [25] and exemplifies a system with a complicated phase diagram.

The fact that the family of local Heisenberg generators $\{L_\Lambda; \Lambda \in I(c)\}$ is also responsible for the classical part of the dynamics is the origin for fundamental mathematical difficulties. In spite of the result that in the folium representation (cf. below definition 3) $\lim_\Lambda \pi_F(L_\Lambda(a))$ exists in the $\sigma(M_F, M_{F*})$-topology for all $a \in A$ and defines an (anti-symmetric) derivation with the domain $\pi_F(A)$, it is not $\sigma(M_F, M_{F*})$-closable to a generator of a W^*-automorphism group in M_F. The convergence of the finite time translations has thus to be investigated by means of the complete power series within selected expectation values (cf. theorem 8 and corollary 9). We have the impression that estimations like those in lemma 6 may not be improved. Thus it seems that this model class is the extreme case of long-range interactions which allow for a well-defined limiting dynamics. In this limiting dynamics infinitely many degrees of freedom influence each other at the same time and bring about a macroscopic change of the system within an infinitesimal time interval.

This would be a physical interpretation of the discontinuity in time of the limiting Heisenberg dynamics. By this effect the existence of generators is prevented and the notion of a global Hamiltonian (and similarly of global generators of symmetry transformations) breaks down, unless the set of observables and states is reduced. On the other hand, these long range interactions are responsible for finite current densities of the extensive observables in the thermodynamic limit, which may flow between the macroscopic subsystems. Just this aspect constitutes a microscopic quantum mechanical foundation of directly observable, macroscopic nonlinear dynamical phenomena.

Since long range interacting models of the mean field type are rather popular, it arises the equestion if the presented existence proofs for the global limiting dynamics have not been done before. This construction, however, is a problem which may be attacked only in the framework of algebraic quantum theory. Thus, the number of reference articles is reduced drastically. The pioneering work of Hepp and Lieb [17] deals only with the classical aspects of the global dynamics. That is, the main point of their investigation (where they do not study reservoirs) is the derivation of an equation like that in our theorem 13. In [16] some arguments concerning the local action of the limiting dynamics in pure phase representations are supplemented. But the necessary estimations are not worked out completely, even in this restricted situation. The global dynamics is also discussed in [26], but the questions concerning the generators are not dealt with correctly. A very general and elegant approach is presented in [27]. It would be interesting to compare the up to now unpublished proofs of the results in this reference with those elaborated here.

Appendix

The proofs give only an outline of the most important steps. The extended step by step formulation may be read in [20].

Proof of Lemma 6

The existence of the uniform bound $M(c) < \infty$ with respect to some lattice subset $I(c)$ we get by the definitions of the volume functions and the concentration dependent absorbing index set $I(c)$. To prove i) and ii) we have to make several inductions.

To i). The first inequality is simply given by the direct sum structure of the unit cell operator x and the morphisms

$$\pi_{(\lambda_q, q)}: B \to A .$$

Now, with a first induction we get for $(\lambda_q, q) \in \Lambda \in I(c)$, with Λ big, the estimation

$$\| L^i_\Lambda(\pi_{(\lambda_q, q)}(x)) \| \leqq \| x \|_\infty \, i! \, M(c)^i .$$

This induction is done in [14] for simple lattices, and in the case of a composite lattice system the proof runs on the same lines if we pay attention to the sublattice structurized permutation automorphism group $\{\alpha_p\}_{p \in P}$.

In the last step we get by induction with respect to $j \in N$ for arbitrary $i \in N$ and all $\lambda \subseteq \chi \subseteq \varkappa \subseteq \Lambda \in I(c)$, with $\varkappa$, Λ big, together with the product formula for derivations:

$$\begin{aligned} &\| L^i_\Lambda(L^{j+1}_\varkappa(\pi_{(\lambda_q, q)}(x))) \| \\ &\leqq \sum_{k=1}^{r} \sum_{n \in v(k)} \Big\{ |a(k,n)| \, \| L^i_\Lambda(L^j_\varkappa(\pi_{(\lambda_q, q)}([e_{n,k}, x]))) \| \\ &+ \sum_{s=1}^{r} \sum_{t \in v(s)} |V(k,n,s,t,\varkappa)| \cdot \left(\sum_{u=0}^{j} \binom{j}{u} \right. \\ &\cdot \| L^i_\Lambda(L^{j-u}_\varkappa(m_\varkappa(e_{n,k}))) \, L^i_\Lambda(L^u_\varkappa(\pi_{(\lambda_q, q)}([e_{t,s}, x]))) \\ &\left. + L^i_\Lambda(L^{j-u}_\varkappa(\pi_{(\lambda_q, q)}([e_{n,k}, x]))) \, L^i_\Lambda(L^i_\varkappa(m_\varkappa(e_{t,s}))) \| \right) \Big\} \\ &\leqq \| x \|_\infty \, M(c)^{(i+j+1)} (i+j+1)! . \end{aligned}$$

We used in the last inequality for $i, j \in \mathbb{N}$ the formula

$$\sum_{q=0}^{i} \sum_{k=0}^{j} \binom{i}{q} \binom{j}{k} (i+j-k-q)! \, (k+q)! = (i+j+1)! ,$$

which is given in [21], together with the induction assumption.

To ii): Let be $\chi \subseteq \Lambda \in I(c)$, then for arbitrary χ-local $a \in A_\chi$ we get

$$\begin{aligned} [m_\Lambda(x), a] &= |\Lambda|^{-1} \sum_{q \in \Lambda} [\pi_q(x), a] \\ &= (|\chi| / |\Lambda|) \, b \end{aligned}$$

with the χ-local operator

$$b := [m_\chi(x), a] \in A_\chi .$$

Now the proof runs with the same induction steps as above. Analogously we get as in [14], Lemma B.2:

$$\| L^i_\Lambda(a) \| \leqq \| a \| \, i! \, (|\chi| \, M(c))^i ,$$

and with this and a similar estimation as in i) we have the result.

Proof of Lemma 7

Because of the uniform boundedness of the local commutators with respect to the lattice regions, the product formula for commutators and the continuity of the product in the strong operator topology on bounded sets, it is sufficient to show whenever $\chi \subseteq \varkappa \subseteq \Lambda \in I(c)$:

i) for arbitrary $x \in B$ and $i, j \in \mathbb{N}$ the operator

$$L_\Lambda^i(L_\varkappa^j(m_\chi(x)))$$

is a generalized mean field polynomial with complex valued coefficients;

ii) for fixed $\chi \in I(c)$ and $a \in A_\chi$ $i, j \in \mathbb{N}$ the operator

$$L_\Lambda^i(L_\varkappa^j(a))$$

is generalized mean field polynomial with coefficients localized in A_χ.

Let be $\chi \subseteq \varkappa \subseteq \Lambda \in I$ and $a \in A_\chi$, then for $i \in \mathbb{N}$ it holds

$$L_\Lambda^{i+1}(a) = \sum_{k=1}^{r} \sum_{n \in v(k)} \Big\{ a(k,n)\,|\varkappa|\, L_\Lambda^i([m_\chi(e_{n,k}), a]) + \sum_{s=1}^{r} \sum_{t \in v(s)} V(k,n,s,t,\Lambda) \sum_{u=0}^{i} \binom{i}{u} |\chi| \cdot (L_\Lambda^{i-u}(m_\Lambda(e_{n,k}))\, L_\Lambda^u([m_\varkappa(e_{t,s}), a]) + L_\Lambda^{i-u}([m_\chi(e_{n,k}), a])\, L_\Lambda^u(m_\Lambda(e_{t,s}))) \Big\}.$$

Now set $a := m_\chi(x)$, $x \in B$, then with the equation

$$|x|\,[m_\chi(e_{n,k}), m_\chi(x)] = m_\chi([e_{n,k}, x])$$

we get the induction for the subproposition i) in the case $j = 0$. Therefore also $L_\Lambda^i(m_\Lambda(x))$ is a generalized mean field polynomial, and it follows ii) for $j = 0$. The induction on $j \in \mathbb{N}$ results from similar transformations (compare the proof of lemma 6). Due to i) and ii) we have shown that for every generalized mean field polynomial $\Gamma(\ldots)$ with localized coefficient operators and for all $i, j \in \mathbb{N}$ the operator

$$L_\Lambda^i(L_\varkappa^j(\Gamma(\Lambda_{1,1}, \ldots, \Lambda_{k,s}; x_1, \ldots, x_s)))$$

is again a generalized mean field polynomial with localized coefficients whenever $\Lambda_{1,1}, \ldots, \Lambda_{n,s} \subseteq \varkappa \subseteq \Lambda$ holds. Combining this with proposition 5 the convergences of the lemma are proven.

Proof of Theorem 8

It is sufficient to prove the convergence properties of the local dynamical transformation for a single mean field $m_\Lambda(x)$ and an arbitrary local $a \in A_\chi$, $\chi \in I$ fixed. Let $w \in \mathbb{R}^+$, then we define

$$K(w) := \left\{ z \in C;\, |z| < \frac{w}{M(c)} \right\},$$

$$G(w) := \left\{ z \in C;\, |\mathrm{Im}(z)| < \frac{w}{M(c)} \right\}.$$

From lemma 6 we get the uniform boundedness of expressions like $\alpha_{\Lambda,t}(\alpha_{\varkappa,s}(m_\chi(x)))$ with respect to $\chi \subseteq \varkappa \subseteq \Lambda \in I(c)$ and $(t,s) \in G(1/2) \times G(1/2)$ within compact subsets. From the lemmas 6 and 7 it follows that the convergence properties in the theorem for $\Gamma(\ldots) := m_\Lambda(x)$ for all $(t,s) \in K(1/2) \times K(1/2)$ are valid.

From the convergence with respect to the strong operator topology in any representation $\{\eta, X\}$ quasi contained in $\{\pi_F, H_F\}$ it results the convergence of the Greens functions of the type

$$\lim_{\Lambda \in I(c)} \lim_{\varkappa \in I(c)} \lim_{\chi \in I(c)} \langle \omega; a\, \alpha_{\Lambda,t}(\alpha_{\varkappa,s}(m_\chi(x)))\, b \rangle = \lim_{\Lambda \in I(c)} \langle \omega; a\, \alpha_{\Lambda,t}(\alpha_{\Lambda,s}(m_\Lambda(x)))\, b \rangle$$

and similar ones for all $(t,s) \in K(1/2) \times K(1/2)$, $a, b \in A$ and $\omega \in F$. Because of the existence of absorbing sequences $\{\Lambda(n,c)\}_{n \in \mathbb{N}}$, $\Lambda(n,c) \in I(c)$ in every index subset $I(c)$ (there is an easy procedure to construct such absorbing lattice index sequences, which is not of interest here). We can use the theory of analytic functions to get the convergence of the Greens function in the region of uniform boundedness. Therefore we get the proposed convergence in the strong operator topology on the von Neumann algebra M_F $(=\pi_F(A)'')$, and since the representation $\{\eta, X\}$ is quasicontained in $\{\pi_F, H_F\}$ we get it on $M_\eta = \pi_\eta(A)''$, too.

The convergence for local operators $a \in A_\chi$, with $\chi \in I$ fixed, can be proven in the same manner by replacing the analyticity and convergence regions $(K(1/2) \to K(1/2\,|\chi|)$ and $G(1/2) \to G(1/2\,|\chi|))$.

Proof of Theorem 10

From corollary 9 we get for every $a \in A$, $\omega \in F$ and $t \in \mathbb{R}$ the limits

$$\langle v_{c,t}(\omega); a \rangle := \lim_{\Lambda \in I(c)} \langle \omega; \alpha_{\Lambda,t}(a) \rangle$$

by approximating a in norm by strictly local operators. Because of the continuous differentiability of the Greens functions with local $a \in A_\chi$, $\chi \in I$, these limits are continuous functions in $t \in \mathbb{R}$. Therefore we have defined for every $t \in \mathbb{R}$

$$v_{c,t} := F \rightarrow S(A)$$

as an affine and positivity preserving map on the folium F. From [11], the Jordan decomposition of functionals on A^*, and the $\sigma(A^*, A)$-density of F in $S(A)$ we get the norm continuity of $v_{c,t}$.

The remaining problem is to prove $v_{c,t}(F) \subseteq F$. Then with the defining equation and the interchangeability of the limits (giving the equations for Greens functions in theorem 8!) we immediately get the group properties.

From permutation invariance of the local Hamiltonians it is easy to find the relation $v_{c,t}(S^P) \subseteq S^P$.

Because of the norm continuity it is sufficient to prove for any $\omega \in S^P$ and $a \in A$, with $\langle \omega; a^* a \rangle = 1$, the inclusion

$$v_{c,t}(\omega_a) \in F, \quad t \in \mathbb{R},$$

with $\omega_a := \langle \omega; a^* a \rangle$.

It is well known that the folium F and the normal states $M_{F,*,+1}$ are affine isomorphic sets, there is a bijective affine and isometric map

$$j\colon F \rightarrow M_{F,*,+1}.$$

Because of $v_{c,t}(\omega) \in S^P$ and theorem 8 the following equation is valid for all $b \in A$:

$$\begin{aligned}\langle v_{c,t}(\omega_a); b \rangle &= \lim_{\Lambda \in I(c)} \langle \omega; a^* \alpha_{\Lambda,t}(b)\, a \rangle \\ &= \lim_{\Lambda \in I(c)} \lim_{\varkappa \in I(c)} \langle \omega; \alpha_{\varkappa,t}(\alpha_{\Lambda,-t}(a^*)\, b\, \alpha_{\Lambda,-t}(a)) \rangle \\ &= \langle j(v_{c,t}(\omega)); s-\lim_{\Lambda \in I(c)} \pi_F(\alpha_{\Lambda,-t}(a^*))\, \pi_F(b) \\ &\qquad \cdot s-\lim_{\Lambda \in I(c)} \pi_F(\alpha_{\Lambda,-t}(a)) \rangle,\end{aligned}$$

and therefore this is a normal functional on M_F Since we also know that $v_{c,t}(\omega_a) \in S(A)$, this really is a state on M_F, and by the bijection $j^{-1}\colon M_{F,*,+1} \rightarrow F$ we get $v_{c,t}(\omega_a) \in F$.

Proof of Theorem 13

Every component $f_{j,q}(m, c)$ is at least a polynomial of order two with respect to $m \in \bigotimes_{q=1}^{r} \mathbb{R}^{n(q)^2}$. Therefore, the Cauchy differential equation has for every $m(0) = w$, $w \in W$, a unique solution for every $t \in \mathbb{R}$.

For arbitrary $x \in B$, $\omega \in S^P$ and $t \in \mathbb{R}$ we define

$$m(v_{c,t}(\omega), x, c) := \lim_{\Lambda \in I(c)} \langle v_{c,t}(\omega); m_\Lambda(x) \rangle,$$

and by parametrization we get

$$m(v_{c,t}(\omega), e_{j,q}, c) = \mathbb{P}(v_{c,t}(\omega))_{j,q}\, c_q$$

for all $j \in v(q)$ and $1 \leqq q \leqq r$. From corollary 9 and the definition of $f(m,c)$ we obtain the equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(v_{c,t}(\omega))_{j,q} &= (c_q)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} m(v_{c,t}(\omega), e_{j,q}, c) \\ &= \left(\frac{i}{c_q}\right) \lim_{\Lambda \in I(c)} \langle v_{c,t}(\omega); [h_\Lambda, m_\Lambda(e_{j,q})] \rangle \\ &= f_{j,q}(\mathbb{P}(v_{c,t}(\omega))),\end{aligned}$$

which shows that $m(t) := \mathbb{P}(v_{c,t}(\omega))$ is a solution of the differential equation with respect to the initial value

$$m(0) := \mathbb{P}(\omega).$$

As we know that every solution of the Cauchy problem is unique we arrive at the assertions of the theorem.

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